

Part III

Intermediate Integration Topics

The 5th Wave

By Rich Tennant



"Okay, ma'am, I'm going to ask you to walk a straight line, then I'm going to ask you to bisect that line with a perpendicular line that slopes to the equation $y = 3x + 5$."

In this part . . .

With the basics of calculating integrals under your belt, the focus becomes using integration as a problem-solving tool. You discover how to solve more complex area problems and how to find the surface area and volume of solids.

Chapter 9

Forging into New Areas: Solving Area Problems

In This Chapter

- ▶ Evaluating improper integrals
 - ▶ Solving area problems with more than one function
 - ▶ Measuring the area between functions
 - ▶ Finding unsigned areas
 - ▶ Understanding the Mean Value Theorem and calculating average value
 - ▶ Figuring out arc length
-

With your toolbox now packed with the *hows* of calculating integrals, this chapter (and Chapter 10) introduces you to some of the *whys* of calculating them.

I start with a simple rule for expressing an area as two separate definite integrals. Then I focus on improper integrals, which are integrals that are either horizontally or vertically infinite. Next, I give you a variety of practical strategies for measuring areas that are bounded by more than one function. I look at measuring areas between functions, and I also get you clear on the distinction between signed area and unsigned area.

After that, I introduce you to the Mean Value Theorem for Integrals, which provides the theoretical basis for calculating average value. Finally, I show you a formula for calculating arc length, which is the exact length between two points along a function.

Breaking Us in Two

Here's a simple but handy rule that looks complicated but is really very easy:

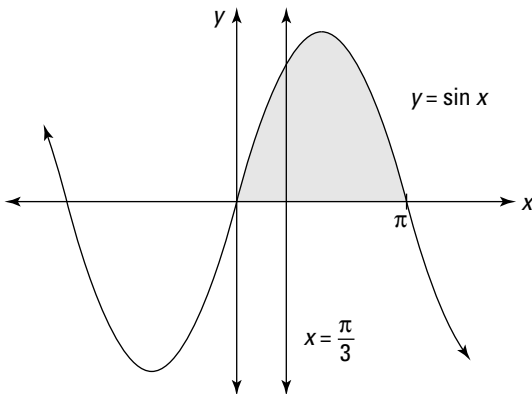
$$\int_a^b f(x) dx = \int_a^n f(x) dx + \int_n^b f(x) dx$$

This rule just says that you can split an area into two pieces, and then add up the pieces to get the area that you started with.

For example, the entire shaded area in Figure 9-1 is represented by the following integral, which you can evaluate easily:

$$\begin{aligned} \int_0^{\pi} \sin x dx \\ &= -\cos x \Big|_{x=0}^{x=\pi} \\ &= -\cos \pi - (-\cos 0) \\ &= 1 + 1 = 2 \end{aligned}$$

Figure 9-1:
Splitting
the area
 $\int_0^{\pi} \sin x dx$
into two
smaller
pieces.



Drawing a vertical line at $x = \frac{\pi}{3}$ and splitting this area into two separate regions results in two separate integrals:

$$\int_0^{\frac{\pi}{3}} \sin x dx + \int_{\frac{\pi}{3}}^{\pi} \sin x dx$$

It should come as no great shock that the sum of these two smaller regions equals the entire area:

$$\begin{aligned}
 &= -\cos x \Big|_{x=0}^{x=\frac{\pi}{3}} + -\cos x \Big|_{x=\frac{\pi}{3}}^{x=\pi} \\
 &= -\cos \frac{\pi}{3} - (-\cos 0) + (-\cos \pi) - (-\cos \frac{\pi}{3}) \\
 &= \cos 0 - \cos \pi \\
 &= 1 + 1 = 2
 \end{aligned}$$

Although this idea is ridiculously simple, splitting an integral into two or more integrals becomes a powerful tool for solving a variety of the area problems in this chapter.

Improper Integrals

Improper integrals come in two varieties — horizontally infinite and vertically infinite:

- ✓ A *horizontally infinite* improper integral contains either ∞ or $-\infty$ (or both) as a limit of integration. See the next section, “Getting horizontal,” for examples of this type of integral.
- ✓ A *vertically infinite* improper integral contains at least one vertical asymptote. I discuss this further in the later section “Going vertical.”

Improper integrals become useful for solving a variety of problems in Chapter 10. They’re also useful for getting a handle on infinite series in Chapter 12. Evaluating an improper integral is a three-step process:

1. **Express the improper integral as the limit of an integral.**
2. **Evaluate the integral by whatever method works.**
3. **Evaluate the limit.**

In this section, I show you, step by step, how to evaluate both types of improper integrals.

Getting horizontal

The first type of improper integral occurs when a definite integral has a limit of integration that’s either ∞ or $-\infty$. This type of improper integral is easy to spot because infinity is right there in the integral itself. You can’t miss it.

For example, suppose that you want to evaluate the following improper integral:

$$\int_1^{\infty} \frac{1}{x^3} dx$$

Here's how you do it, step by step:

1. Express the improper integral as the limit of an integral.

When the upper limit of integration is ∞ , use this equation:

$$\int_a^{\infty} f(x) dx = \lim_{c \rightarrow \infty} \int_a^c f(x) dx$$

So here's what you do:

$$\int_1^{\infty} \frac{1}{x^3} dx = \lim_{c \rightarrow \infty} \int_1^c \frac{1}{x^3} dx$$

2. Evaluate the integral:

$$\begin{aligned} \lim_{c \rightarrow \infty} \left(-\frac{1}{2x^2} \Big|_{x=1}^{x=c} \right) \\ = \lim_{c \rightarrow \infty} \left(-\frac{1}{2c^2} + \frac{1}{2} \right) \end{aligned}$$

3. Evaluate the limit:

$$= \frac{1}{2}$$

Before moving on, reflect for one moment that the area under an *infinitely* long curve is actually *finite*. Ah, the magic and power of calculus!

Similarly, suppose that you want to evaluate the following:

$$\int_{-\infty}^0 e^{5x} dx$$

Here's how you do it:

1. Express the integral as the limit of an integral.

When the lower limit of integration is $-\infty$, use this equation:

$$\int_{-\infty}^b f(x) dx = \lim_{c \rightarrow -\infty} \int_c^b f(x) dx$$

So here's what you write:

$$\int_{-\infty}^0 e^{5x} dx = \lim_{c \rightarrow -\infty} \int_c^0 e^{5x} dx$$

2. Evaluate the integral:

$$\begin{aligned}
 &= \lim_{c \rightarrow -\infty} \left(\frac{1}{5} e^{5x} \Big|_{x=c}^{x=0} \right) \\
 &= \lim_{c \rightarrow -\infty} \left(\frac{1}{5} e^0 - \frac{1}{5} e^{5c} \right) \\
 &= \lim_{c \rightarrow -\infty} \left(\frac{1}{5} - \frac{1}{5} e^{5c} \right)
 \end{aligned}$$

3. Evaluate the limit — in this case, as c approaches $-\infty$, the first term is unaffected and the second term approaches 0:

$$= \frac{1}{5}$$

Again, calculus tells you that, in this case, the area under an infinitely long curve is finite.

Of course, sometimes the area under an infinitely long curve is infinite. In these cases, the improper integral cannot be evaluated because the limit does not exist (DNE). Here's a quick example that illustrates this situation:

$$\int_1^{\infty} \frac{1}{x} dx$$

It may not be obvious that this improper integral represents an infinitely large area. After all, the value of the function approaches 0 as x increases. But watch how this evaluation plays out:

1. Express the improper integral as the limit of an integral:

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{c \rightarrow \infty} \int_1^c \frac{1}{x} dx$$

2. Evaluate the integral:

$$\begin{aligned}
 &= \lim_{c \rightarrow \infty} \ln x \Big|_{x=1}^{x=c} \\
 &= \lim_{c \rightarrow \infty} \ln c - \ln 1
 \end{aligned}$$

At this point, you can see that the limit explodes to infinity, so it doesn't exist. Therefore, the improper integral can't be evaluated, because the area that it represents is infinite.

Going vertical

Vertically infinite improper integrals are harder to recognize than those that are horizontally infinite. An integral of this type contains at least one vertical asymptote in the area that you're measuring. (A *vertical asymptote* is a value

of x where $f(x)$ equals either ∞ or $-\infty$. See Chapter 2 for more on asymptotes.) The asymptote may be a limit of integration or it may fall someplace between the two limits of integration.



Don't try to slide by and evaluate improper integrals as proper integrals. In most cases, you'll get the wrong answer!

In this section, I show you how to handle both cases of vertically infinite improper integrals.

Handling asymptotic limits of integration

Suppose that you want to evaluate the following integral:

$$\int_0^1 \frac{1}{\sqrt{x}} dx$$

At first glance, you may be tempted to evaluate this as a proper integral. But this function has an asymptote at $x = 0$. The presence of an asymptote at one of the limits of integration forces you to evaluate this one as an improper integral:

1. Express the integral as the limit of an integral:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} \int_c^1 \frac{1}{\sqrt{x}} dx$$

Notice that in this limit, c approaches 0 from the right — that is, from the positive side — because this is the direction of approach from inside the limits of integration. (That's what the little plus sign ($^+$) in the limit in Step 2 means.)

2. Evaluate the integral:

This integral is easily evaluated as $x^{-\frac{1}{2}}$, using the Power Rule as I show you in Chapter 4, so I spare you the details here:

$$= \lim_{c \rightarrow 0^+} 2\sqrt{x} \Big|_{x=c}^{x=1}$$

3. Evaluate the limit:

$$= \lim_{c \rightarrow 0^+} 2\sqrt{1} - 2\sqrt{c}$$

At this point, direct substitution provides you with your final answer:

$$= 2$$

Piecing together discontinuous integrands

In Chapter 3, I discuss the link between integrability and continuity: If a function is continuous on an interval, it's also integrable on that interval. (Flip to Chapter 3 for a refresher on this concept.)

Some integrals that are vertically infinite have asymptotes not at the edges but someplace in the middle. The result is a *discontinuous integrand* — that is, function with a discontinuity on the interval that you're trying to integrate.



Discontinuous integrands are the trickiest improper integrals to spot — you really need to know how the graph of the function that you're integrating behaves. (See Chapter 2 to see graphs of the elementary functions.)

To evaluate an improper integral of this type, separate it at each asymptote into two or more integrals, as I demonstrate earlier in this chapter in “Breaking Us in Two.” Then evaluate each of the resulting integrals as an improper integral, as I show you in the previous section.

For example, suppose that you want to evaluate the following integral:

$$\int_0^{\pi} \sec^2 x \, dx$$

Because the graph of $\sec x$ contains an asymptote at $x = \frac{\pi}{2}$ (see Chapter 2 for a view of this graph), the graph of $\sec^2 x$ has an asymptote in the same place, as you see in Figure 9-2.

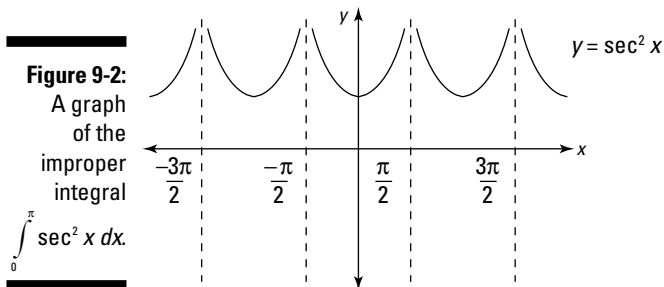


Figure 9-2:
A graph
of the
improper
integral

$$\int_0^{\pi} \sec^2 x \, dx.$$

To evaluate this integral, break it into two integrals at the value of x where the asymptote is located:

$$\int_0^{\pi} \sec^2 x \, dx = \int_0^{\frac{\pi}{2}} \sec^2 x \, dx + \int_{\frac{\pi}{2}}^{\pi} \sec^2 x \, dx$$

Now, evaluate the sum of the two resulting integrals.



You can save yourself a lot of work by noticing when two regions are symmetrical. In this case, the asymptote at $x = \frac{\pi}{2}$ splits the shaded area into two symmetrical regions. So you can find one integral and then double it to get your answer:

$$= 2 \int_0^{\frac{\pi}{2}} \sec^2 x \, dx$$

Now, use the steps from the previous section to evaluate this integral:

1. Express the integral as the limit of an integral:

$$= 2 \lim_{c \rightarrow \frac{\pi}{2}^-} \int_0^c \sec^2 x \, dx$$

In this case, the vertical asymptote is at the upper limit of integration, so c approaches $\frac{\pi}{2}$ from the left — that is, from inside the interval where you're measuring the area.

2. Evaluate the integral:

$$\begin{aligned} &= 2 \lim_{c \rightarrow \frac{\pi}{2}^-} \left(\tan x \Big|_{x=0}^{x=c} \right) \\ &= 2 \lim_{c \rightarrow \frac{\pi}{2}^-} (\tan c - \tan 0) \end{aligned}$$

3. Evaluate the limit:

Note that $\tan \frac{\pi}{2}$ is undefined, because the function $\tan x$ has an asymptote at $x = \frac{\pi}{2}$, so the limit does not exist (DNE). Therefore, the integral that you're trying to evaluate also does not exist because the area that it represents is infinite.

Solving Area Problems with More Than One Function

The definite integral allows you to find the signed area under any interval of a single function. But when you want to find an area defined by more than one function, you need to be creative and piece together a solution. Professors love these problems as exam questions, because they test your reasoning skills as well as your calculus knowledge.

Fortunately, when you approach problems of this type correctly, you find that they're not terribly difficult. The trick is to break down the problem into two or more regions that you can measure by using the definite integral, and then use addition or subtraction to find the area that you're looking for.

In this section, I get you up to speed on problems that involve more than one definite integral.

Finding the area under more than one function

Sometimes, a single geometric area is described by more than one function. For example, suppose that you want to find the shaded area shown in Figure 9-3.

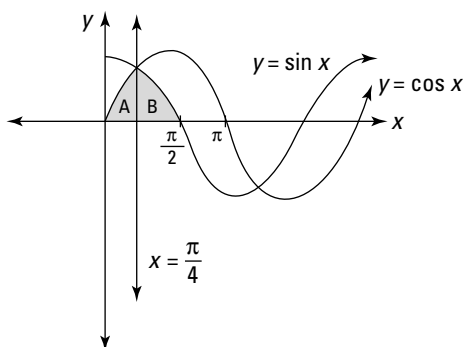


Figure 9-3:
Finding the
area under
 $y = \sin x$ and
 $y = \cos x$
from 0 to $\frac{\pi}{2}$.

The first thing to notice is that the shaded area isn't under a single function, so you can't expect to use a single integral to find it. Instead, the region labeled A is under $y = \sin x$ and the region labeled B is under $y = \cos x$. First, set up an integral to find the area of both of these regions:

$$A = \int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$B = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

Now, set up an equation to find their combined area:

$$A + B = \int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

At this point, you can evaluate each of these integrals separately. But there's an easier way.



Because region A and region B are symmetrical, they have the same area. So you can find their combined area by doubling the area of a single region:

$$=2A = 2 \int_0^{\frac{\pi}{4}} \sin x \, dx$$

I choose to double region A because the integral limits of integration are easier, but doubling region B also works. Now, integrate to find your answer:

$$\begin{aligned} &= 2(-\cos x) \Big|_{x=0}^{x=\frac{\pi}{4}} \\ &= 2\left(-\cos \frac{\pi}{4} - (-\cos 0)\right) \\ &= 2\left(-\frac{\sqrt{2}}{2} + 1\right) \\ &= 2 - \sqrt{2} \approx 0.586 \end{aligned}$$

Finding the area between two functions

To find an area between two functions, you need to set up an equation with a combination of definite integrals of both functions. For example, suppose that you want to calculate the shaded area in Figure 9-4.

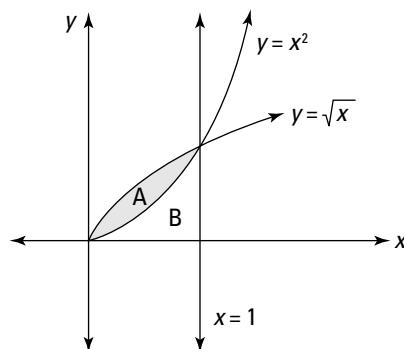


Figure 9-4:
Finding the area between $y = x^2$ and $y = \sqrt{x}$.

First, notice that the two functions $y = x^2$ and $y = \sqrt{x}$ intersect where $x = 1$. This is important information because it enables you to set up two definite integrals to help you find region A:

$$A + B = \int_0^1 \sqrt{x} \, dx$$

$$B = \int_0^1 x^2 \, dx$$

Although neither equation gives you the exact information that you're looking for, together they help you out. Just subtract the second equation from the first as follows:

$$A = A + B - B = \int_0^1 \sqrt{x} \, dx - \int_0^1 x^2 \, dx$$

With the problem set up properly, now all you have to do is evaluate the two integrals:

$$\begin{aligned} &= \left(\frac{2}{3} x^{\frac{3}{2}} \Big|_{x=0}^{x=1} \right) - \left(-\frac{1}{3} x^3 \Big|_{x=0}^{x=1} \right) \\ &= \left(\frac{2}{3} - 0 \right) - \left(\frac{1}{3} - 0 \right) = \frac{1}{3} \end{aligned}$$

So the area between the two curves is $\frac{1}{3}$.

As another example, suppose that you want to find the shaded area in Figure 9-5.

This time, the shaded area is two separate regions, labeled A and B. Region A is bounded above by $y = x^{\frac{1}{3}}$ and bounded below by $y = x$. However, for region B, the situation is reversed, and the region is bounded above by $y = x$ and bounded below by $y = x^{\frac{1}{3}}$. I also label region C and region D, both of which figure into the problem.

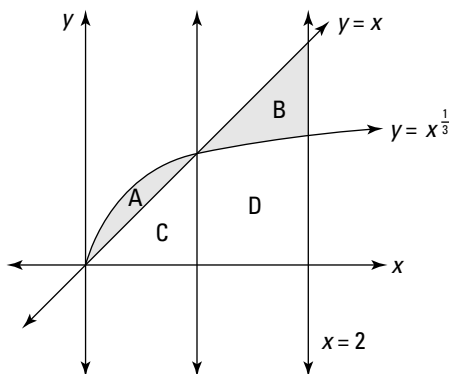


Figure 9-5:
Finding the
area
between
 $y = x$ and
 $y = x^{\frac{1}{3}}$.

The first important step is finding where the two functions intersect — that is, where the following equation is true:

$$x = x^{\frac{1}{3}}$$

Fortunately, it's easy to see that $x = 1$ satisfies this equation.

Now, you want to build a few definite integrals to help you find the areas of region A and region B. Here are two that can help with region A:

$$A + C = \int_0^1 x^{\frac{1}{3}} dx$$

$$C = \int_0^1 x dx = \frac{1}{2}$$

Notice that I evaluate the second definite integral *without* calculus, using simple geometry as I show you in Chapter 1. This is perfectly valid and a great timesaver.

Subtracting the second equation from the first provides an equation for the area of region A:

$$A = A + C - C = \int_0^1 x^{\frac{1}{3}} dx - \frac{1}{2}$$

Now, build two definite integrals to help you find the area of region B:

$$B + D = \int_1^2 x dx = \frac{3}{2}$$

$$D = \int_1^2 x^{\frac{1}{3}} dx$$

This time, I evaluate the first definite integral by using geometry instead of calculus. Subtracting the second equation from the first gives an equation for the area of region B:

$$B = B + D - D = \frac{3}{2} - \int_1^2 x^{\frac{1}{3}} dx$$

Now you can set up an equation to solve the problem:

$$\begin{aligned} A + B &= \int_0^1 x^{\frac{1}{3}} dx - \frac{1}{2} + \frac{3}{2} - \int_1^2 x^{\frac{1}{3}} dx \\ &= \int_0^1 x^{\frac{1}{3}} dx - \int_1^2 x^{\frac{1}{3}} dx + 1 \end{aligned}$$

At this point, you're forced to do some calculus:

$$\begin{aligned} &= \left(\frac{3}{4} x^{\frac{4}{3}} \Big|_{x=0}^{x=1} \right) - \left(\frac{3}{4} x^{\frac{4}{3}} \Big|_1^2 + 1 \right) \\ &= \left(\frac{3}{4} (1)^{\frac{4}{3}} - 0 \right) - \left(\frac{3}{4} (2)^{\frac{4}{3}} - \frac{3}{4} (1)^{\frac{4}{3}} \right) + 1 \end{aligned}$$

The rest is just arithmetic:

$$\begin{aligned} &= \frac{3}{4} - \frac{3}{4} (16)^{\frac{1}{3}} + \frac{3}{4} + 1 \\ &= \frac{5}{2} - \frac{3}{4} (16)^{\frac{1}{3}} \\ &\approx 0.6101 \end{aligned}$$

Looking for a sign

The solution to a definite integral gives you the *signed* area of a region (see Chapter 3 for more). In some cases, signed area is what you want, but in some problems you're looking for *unsigned* area.

The signed area above the x -axis is positive, but signed area below the x -axis is negative. In contrast, unsigned area is *always* positive. The concept of unsigned area is similar to the concept of absolute value. So, if it's helpful, think of unsigned area as the absolute value of a definite integral.



In problems where you're asked to find the area of a shaded region on a graph, you're looking for unsigned area. But if you're unsure whether a question is asking you to find signed or unsigned area, ask the professor. This goes double if an exam question is unclear. Most professors will answer clarifying questions, so don't be shy to ask.

For example, suppose that you're asked to calculate the shaded unsigned area that's shown in Figure 9-6.

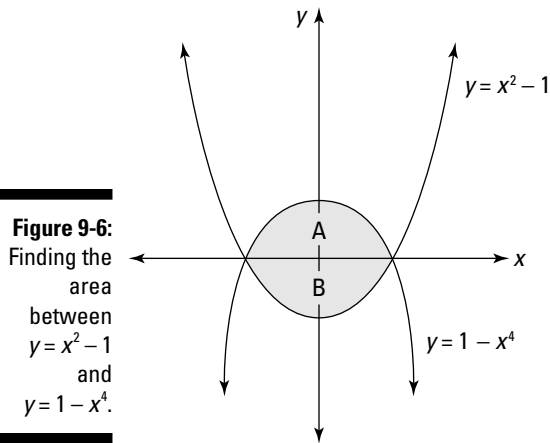


Figure 9-6:
Finding the
area
between
 $y = x^2 - 1$
and
 $y = 1 - x^4$.

This area is actually the sum of region A, which is above the x -axis, and region B, which is below it. To solve the problem, you need to find the sum of the unsigned areas of these two regions.

Fortunately, both functions intersect each other and the x -axis at the same two values of x : $x = -1$ and $x = 1$. Set up definite integrals to find the area of each region as follows:

$$A = \int_{-1}^1 (1 - x^4) dx$$

$$B = - \int_{-1}^1 (x^2 - 1) dx$$

Notice that I negate the definite integral for region B to account for the fact that the definite integral produces negative area below the x -axis. Now, just add the two equations together:

$$A + B = \int_{-1}^1 (1 - x^4) dx - \int_{-1}^1 (x^2 - 1) dx$$

Solving this equation gives you the answer that you're looking for (be careful with all those minus signs!):

$$\begin{aligned} &= \left(x - \frac{1}{5} x^5 \Big|_{x=-1}^{x=1} \right) - \left(\frac{1}{3} x^3 - x \Big|_{x=-1}^{x=1} \right) \\ &= \left[\left(1 - \frac{1}{5} \right) - \left(-1 - -\frac{1}{5} \right) \right] - \left[\left(\frac{1}{3} - 1 \right) - \left(-\frac{1}{3} - -1 \right) \right] \\ &= \frac{4}{5} + \frac{4}{5} - \left(-\frac{2}{3} - \frac{2}{3} \right) \end{aligned}$$

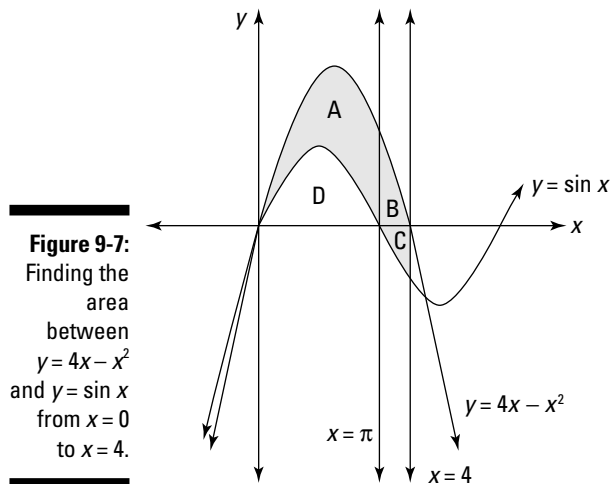
Notice at this point that the expression in the parentheses — representing the signed area of region B — is negative. But the minus sign outside the parentheses automatically flips the sign as intended:

$$= \frac{8}{5} + \frac{4}{3} = \frac{44}{15}$$

Measuring unsigned area between curves with a quick trick

After you understand the concept of measuring unsigned area (which I discuss in the previous section), you're ready for a trick that makes measuring the area between curves very straightforward. As I say earlier in this chapter, professors love to stick these types of problem on exams. So here's a difficult exam question that's worth spending some time with:

Find the unsigned shaded area in Figure 9-7. Approximate your answer to two decimal places by using $\cos 4 = -0.65$.



The first step is to find an equation for the solution (which will probably give you partial credit), and then worry about solving it.

I split the shaded area into three regions labeled A, B, and C. I also label region D, which you need to consider. Notice that $x = \pi$ separates regions A and B, and the x -axis separates regions B and C.

You could find three separate equations for regions A, B, and C, but there's a better way.



To measure the unsigned area between two functions, use this quick trick:

$$\text{Area} = \text{Integral of Top Function} - \text{Integral of Bottom Function}$$

That's it! Instead of measuring the area above and below the x -axis, just plug the two integrals into this formula. In this problem, the top function is $4x - x^2$ and the bottom function is $\sin x$:

$$= \int_0^4 (4x - x^2) dx - \int_0^4 \sin x dx$$

This evaluation isn't too horrible:

$$\begin{aligned} & \left(2x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=4} \right) - \left(-\cos x \Big|_{x=0}^{x=4} \right) \\ &= \left[\left(2(4)^2 - \frac{1}{3}(4)^3 \right) - 0 \right] - (-\cos 4 - -\cos 0) \\ &= 32 - \frac{64}{3} + \cos 4 - 1 \end{aligned}$$

When you get to this point, you can already see that you're on track, because the professor was nice enough to give you an approximate value for $\cos 4$:

$$\approx 32 - 21.33 - 0.65 - 1 = 9.02$$

So the unsigned area between the two functions is approximately 9.02 units.



If the two functions change positions — that is, the top becomes the bottom and the bottom becomes the top — you may need to break the problem up into regions, as I show you earlier in this chapter. But even in this case, you can still save a lot of time by using this trick.

For example, earlier in this chapter, in “Finding the area between two functions,” I measure the shaded area from Figure 9-5 by using four separate regions. Here's how to do it using the trick in this section.

Notice that the two functions cross at $x = 1$. So, from 0 to 1, the top function is $x^{\frac{1}{3}}$ and from 1 to 2 the top function is x . So, set up two separate equations, one for region A and another for the region B:

$$A = \int_0^1 x^{\frac{1}{3}} dx - \int_0^1 x dx$$

$$B = \int_1^2 x dx - \int_1^2 x^{\frac{1}{3}} dx$$

When the calculations are complete, you get the following values for A and B:

$$A = \frac{1}{4}$$

$$B = \frac{9}{4} - \frac{3}{4}(16)^{\frac{1}{3}}$$

Add these two values together to get your answer:

$$A + B = \frac{5}{2} - \frac{3}{4}(16)^{\frac{1}{3}} \approx 0.6101$$

As you can see, the top-and-bottom trick gets you the same answer much more simply than measuring regions.

The Mean Value Theorem for Integrals

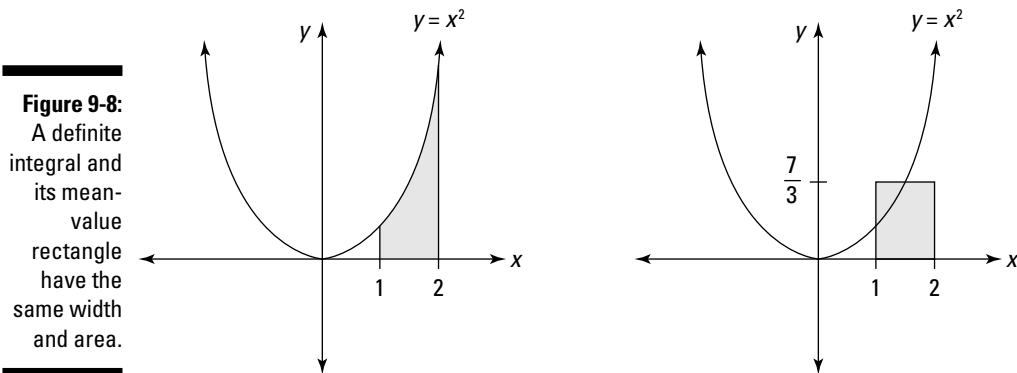
The *Mean Value Theorem for Integrals* guarantees that for every definite integral, a rectangle with the same area and width exists. Moreover, if you superimpose this rectangle on the definite integral, the top of the rectangle intersects the function. This rectangle, by the way, is called the *mean-value rectangle* for that definite integral. Its existence allows you to calculate the *average value* of the definite integral.



Calculus boasts *two* Mean Value Theorems — one for derivatives and one for integrals. This section discusses the Mean Value Theorem for Integrals. You can find out about the Mean Value Theorem for Derivatives in *Calculus For Dummies* by Mark Ryan (Wiley).

The best way to see how this theorem works is with a visual example. The first graph in Figure 9-8 shows the region described by the definite integral

$A = \int_0^1 x^{\frac{1}{3}} dx - \int_0^1 x dx$. This region obviously has a width of 1, and you can evaluate it easily to show that its area is $\frac{7}{3}$.



The second graph in Figure 9-8 shows a rectangle with a width of 1 and an area of $\frac{7}{3}$. It should come as no surprise that this rectangle's height is also $\frac{7}{3}$, so the top of this rectangle intersects the original function.

The fact that the top of the mean-value rectangle intersects the function is mostly a matter of common sense. After all, the height of this rectangle represents the average value that the function attains over a given interval. This value must fall someplace between the function's maximum and minimum values on that interval.

Here's the formal statement of the Mean Value Theorem for Integrals:

If $f(x)$ is a continuous function on the closed interval $[a, b]$, then there exists a number c in that interval such that:

$$\int_a^b f(x) dx = f(c) \cdot (b - a)$$

This equation may look complicated, but it's basically a restatement of this familiar equation for the area of a rectangle:

$$\text{Area} = \text{Height} \cdot \text{Width}$$

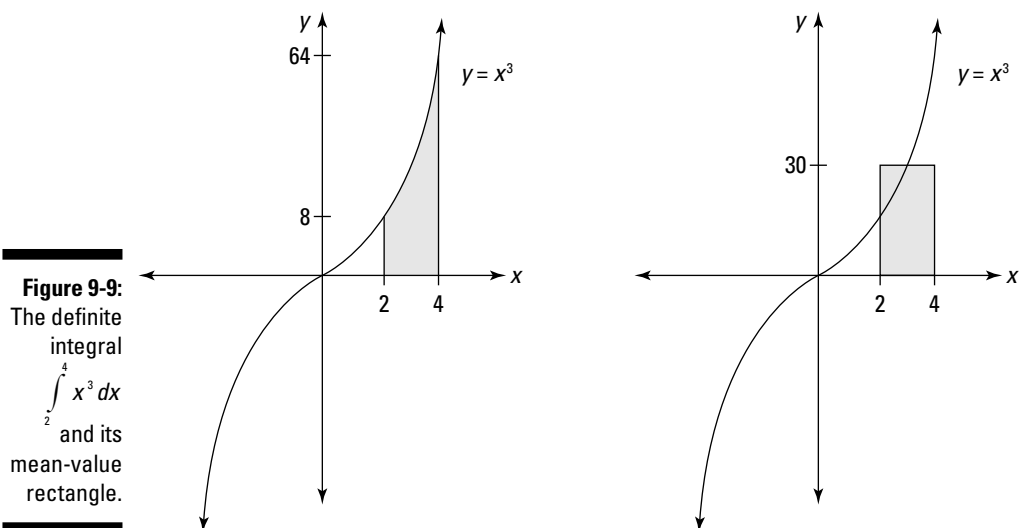
In other words: Start with a definite integral that expresses an area, and then draw a rectangle of equal area with the same width ($b - a$). The height of that rectangle — $f(c)$ — is such that its top edge intersects the function where $x = c$.

The value $f(c)$ is the *average value* of $f(x)$ over the interval $[a, b]$. You can calculate it by rearranging the equation stated in the theorem:

$$f(c) = \frac{1}{b-a} \cdot \int_a^b f(x) dx$$

For example, here's how you calculate the average value of the shaded area in Figure 9-9:

$$\begin{aligned} f(c) &= \frac{1}{4-2} \cdot \int_2^4 x^3 dx \\ &= \frac{1}{2} \left(\frac{1}{4} x^4 \Big|_{x=2}^{x=4} \right) \\ &= \frac{1}{2} \left(\frac{1}{4} 4^4 - \frac{1}{4} 2^4 \right) \\ &= \frac{1}{2} (64 - 4) = 30 \end{aligned}$$



Not surprisingly, the average value of this integral is 30, a value between the function's minimum of 8 and its maximum of 64.

Calculating Arc Length

The arc length of a function on a given interval is the length from the starting point to the ending point as measured along the graph of that function.

In a sense, arc length is similar to the practical measurement of driving distance. For example, you may live only 5 miles from work “as the crow flies,” but when you check your odometer, you may find that the actual drive is

closer to 7 miles. Similarly, the straight-line distance between two points is always less than the arc length along a curved function that connects them.

Using the formula, however, often involves trig substitution (see Chapter 7 for a refresher on this method of integration).

The formula for the arc length along a function $y = f(x)$ from a to b is as follows:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

For example, suppose that you want to calculate the arc length along the function $y = x^2$ from the point where $x = 0$ to the point where $x = 2$ (see Figure 9-10).

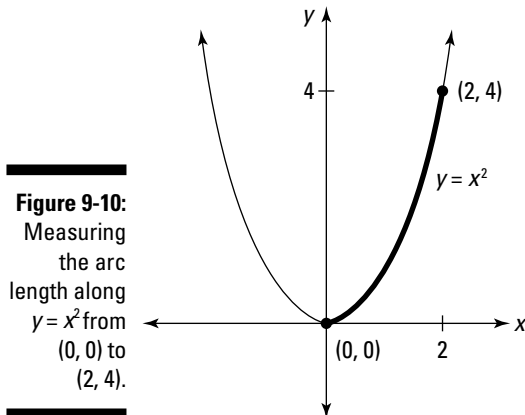


Figure 9-10:
Measuring
the arc
length along
 $y = x^2$ from
(0, 0) to
(2, 4).

Before you begin, notice that if you draw a straight line between these two points, (0, 0) and (2, 4), its length is $\sqrt{20} \approx 4.4721$. So the arc length should be slightly greater.

To calculate the arc length, first find the derivative of the function x^2 :

$$\frac{dy}{dx} = 2x$$

Now, plug this derivative and the limits of integration into the formula as follows:

$$\begin{aligned} & \int_0^2 \sqrt{1 + (2x)^2} dx \\ &= \int_0^2 \sqrt{1 + 4x^2} dx \end{aligned}$$

Calculating arc length usually gives you an opportunity to practice trig substitution — in particular, the tangent case. When you draw your trig substitution triangle, place $\sqrt{1+4x^2}$ on the hypotenuse, $2x$ on the opposite side, and 1 on the adjacent side. This gives you the following substitutions:

$$\sqrt{1+4x^2} = \sec \theta$$

$$2x = \tan \theta$$

$$x = \frac{1}{2} \tan \theta$$

$$dx = \frac{1}{2} \sec^2 \theta d\theta$$

The result is this integral:

$$\frac{1}{2} \int \sec^3 \theta d\theta$$

Notice that I remove the limits of integration because I plan to change the variable back to x before computing the definite integral. I spare you the details of calculating this indefinite integral, but you can see them in Chapter 7. Here's the result:

$$= \frac{1}{4} (\ln |\sec \theta + \tan \theta| + \tan \theta \sec \theta) + C$$

Now, write the each $\sec \theta$ and $\tan \theta$ in terms of x :

$$\frac{1}{4} (\ln |\sqrt{1+4x^2+x^2} + 2x| + 2x \sqrt{1+4x^2}) + C$$

At this point, I'm ready to evaluate the definite integral that I leave off earlier:

$$\begin{aligned} & \int_0^2 \sqrt{1+4x^2} dx \\ &= \frac{1}{4} (\ln |\sqrt{1+4x^2} + 2x| + 2x \sqrt{1+4x^2}) \Big|_{x=0}^{x=2} \\ &= \frac{1}{4} (\ln |\sqrt{1+4(2)^2} + 2(2)| + 2(2)\sqrt{1+4(2)^2}) - 0 \end{aligned}$$

You can either take my word that the second part of this substitution works out to 0 or calculate it yourself. To finish up:

$$\begin{aligned} &= \frac{1}{4} \ln |\sqrt{17} + 4| + \sqrt{17} \\ &\approx 0.5236 + 4.1231 = 4.6467 \end{aligned}$$

Chapter 10

Pump up the Volume: Using Calculus to Solve 3-D Problems

In This Chapter

- ▶ Understanding the meat-slicer method for finding volume
 - ▶ Using inverses to make a problem easier to solve
 - ▶ Solving problems with solids of revolution and surfaces of revolution
 - ▶ Finding the space between two surfaces
 - ▶ Understanding the shell method for finding volume
-

In Chapter 9, I show you a bunch of different ways to use integrals to find area. In this chapter, you add a dimension by discovering how to use integrals to find volumes and surface areas of solids.

First, I show you how to find the volume of a solid by using the meat-slicer method, which is really a 3-D extension of the basic integration tactic you already know from Chapter 1: slicing an area into an infinite number of pieces and adding them up.

As with a real meat slicer, this method works best when the blade is slicing vertically — that is, perpendicular to the x -axis. So, I also show you how to use inverses to rotate some solids into the proper position.

After that, I show you how to solve two common types of problems that calculus teachers just love: finding the volume of a solid of revolution and finding the area of a surface of revolution.

With these techniques in your back pocket, you move on to more complex problems, where a solid is described as the space between two surfaces. These problems are the 3-D equivalent of finding an area between two curves, which I discuss in Chapter 9.

To finish up, I give you an additional way to find the volume of a solid: the shell method. Then, I provide some practical perspective on all the methods in the chapter so you know when to use them.

Slicing Your Way to Success

Did you ever marvel at the way in which a meat slicer turns an entire salami into dozens of tasty little paper-thin circles? Even if you're a vegetarian, calculus provides you with an animal-friendly alternative: the meat-slicer method for measuring the volume of solids.

The *meat-slicer method* works best with solids that have similar cross sections. (I discuss this further in the following section.) Here's the plan:

- 1. Find an expression that represents the area of a random cross section of the solid in terms of x .**
- 2. Use this expression to build a definite integral (in terms of dx) that represents the volume of the solid.**
- 3. Evaluate this integral.**

Don't worry if these steps don't make a whole lot of sense yet. In this section, I show you when and how to use the meat-slicer method to find volumes that would be difficult or impossible without calculus.

Finding the volume of a solid with congruent cross sections

Before I get into calculus, I want to provide a little bit of background on finding the volume of solids. Spending a few minutes thinking about how volume is measured *without* calculus pays off big-time when you step into the calculus arena. This is strictly no-brainer stuff — some basic, solid geometry that you probably know already. So just lie back and coast through this section.

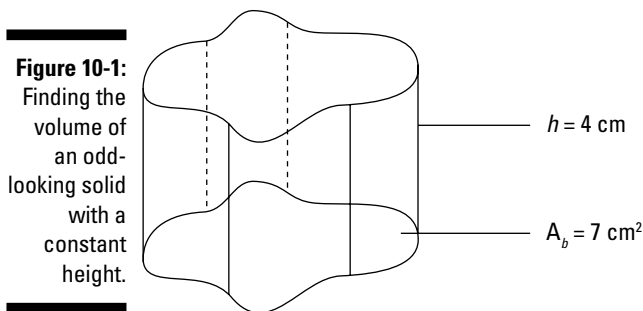
One of the simplest solids to find the volume of is a prism. A *prism* is a solid that has all congruent cross sections in the shape of a polygon. That is, no matter how you slice a prism parallel to its base, its cross section is the same shape and area as the base itself.

The formula for the volume of a prism is simply the area of the base times the height:

$$V = A_b \cdot h$$

So, if you have a triangular prism with a height of 3 inches and a base area of 2 square inches, its volume is 6 cubic inches.

This formula also works for cylinders — which are sort of prisms with a circular base — and generally any solid that has congruent cross sections. For example, the odd-looking solid in Figure 10-1 fits the bill nicely. In this case, you're given the information that the area of the base is 7 cm^2 and the height is 4 cm, so the volume of this solid is 28 cm^3 .



Finding the volume of a solid with congruent cross sections is always simple as long as you know two things:

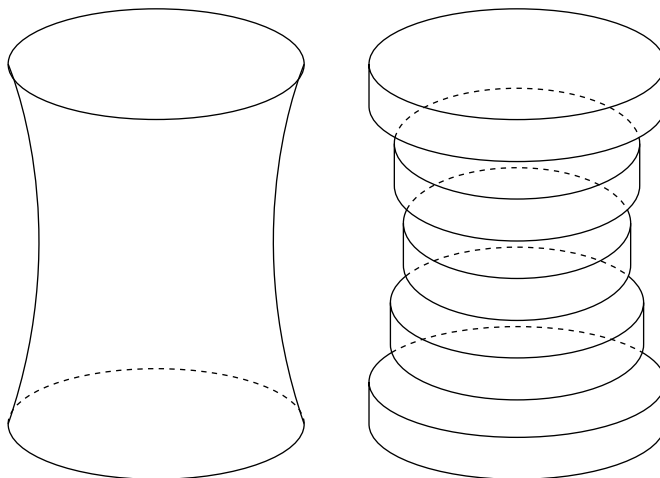
- ✓ The area of the base — that is, the area of any cross section
- ✓ The height of the solid

Finding the volume of a solid with similar cross sections

In the previous section, you didn't have to use any calculus brain cells. But now, suppose that you want to find the volume of the scary-looking hyperbolic cooling tower on the left side of Figure 10-2.

What makes this problem out of the reach of the formula for prisms and cylinders? In this case, slicing parallel to the base always results in the same shape — a circle — but the area may differ. That is, the solid has *similar* cross sections rather than congruent ones.

Figure 10-2:
Estimating
the volume
of a
hyperbolic
cooling
tower by
slicing it into
cylindrical
sections.



You can estimate this volume by slicing the solid into numerous cylinders, finding the volume of each cylinder by using the formula for constant-height solids, and adding these separate volumes. Of course, making more slices improves your estimate. And, as you may already suspect, adding the limit of an infinite number of slices gives you the exact volume of the solid.

Hmmm . . . this is beginning to sound like a job for calculus. In fact, what I hint at in this section is the meat-slicer method, which works well for measuring solids that have similar cross sections.



When a problem asks you to find the volume of a solid, look at the picture of this solid and figure out how to slice it up so that all the cross sections are similar. This is a good first step in understanding the problem so that you can solve it.



To measure weird-shaped solids that *don't* have similar cross sections, you need multivariable calculus, which is the subject of Calculus III. See Chapter 14 for an overview of this topic.

Measuring the volume of a pyramid

Suppose that you want to find the volume of a pyramid with a 6-x-6-unit square base and a height of 3 units. Geometry tells you that you can use the following formula:

$$V = \frac{1}{3}bh = \frac{1}{3}(36)(3) = 36$$

This formula works just fine, but it doesn't give you insight into how to solve similar problems; it works only for pyramids. The meat-slicer method, however, provides an approach to the problem that you can generalize to use for many other types of solids.

To start out, I skewer this pyramid on the x -axis of a graph, as shown in Figure 10-3. Notice that the vertex of the pyramid is at the origin, and the center of the base is at the point $(6, 0)$.

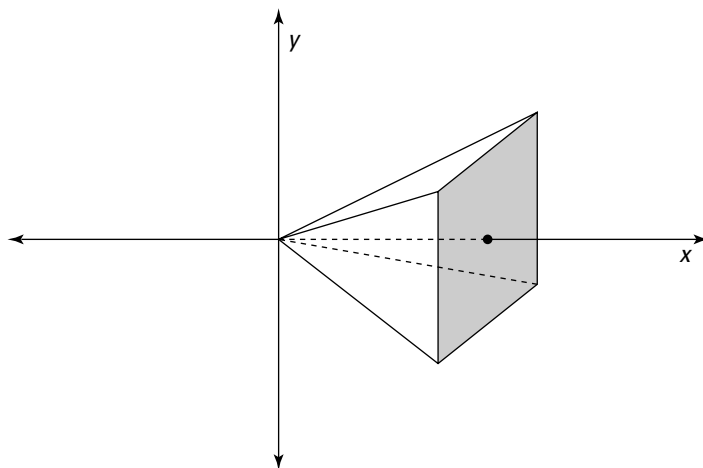


Figure 10-3:
A pyramid
skewered
on a graph
and sliced
into three
pieces.

To find the exact volume of the pyramid, here's what you do:

1. Find an expression that represents the area of a random cross section of the pyramid in terms of x .

At $x = 1$, the cross section is $2^2 = 4$. At $x = 2$, it's $4^2 = 16$. And at $x = 3$, it's $6^2 = 36$. So generally speaking, the area of the cross section is:

$$A = (2x)^2 = 4x^2$$

2. Use this expression to build a definite integral that represents the volume of the pyramid.

In this case, the limits of integration are 0 and 3, so:

$$V = \int_0^3 4x^2 dx$$

3. Evaluate this integral:

$$\begin{aligned} & \frac{4}{3} x^3 \Big|_{x=0}^{x=3} \\ &= \frac{4}{3} 3^3 - 0 = 36 \end{aligned}$$

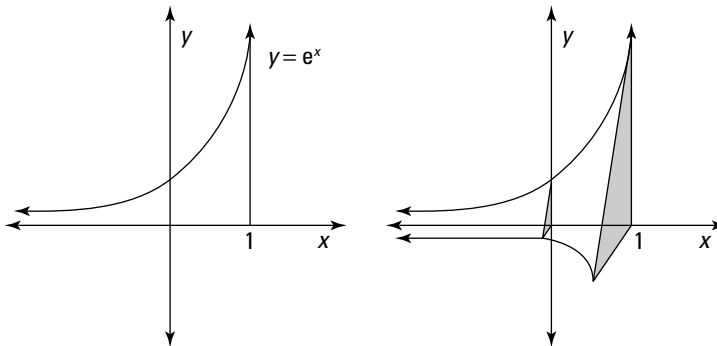
This is the same answer provided by the formula for the pyramid. But this method can be applied to a far wider variety of solids.

Measuring the volume of a weird solid

After you know the basic meat-slicer technique, you can apply it to any solid with a cross section that's a function of x . In some cases, these solids are harder to describe than they are to measure. For example, have a look at Figure 10-4.

The solid in Figure 10-4 consists of two exponential curves — one described by the equation $y = e^x$, and the other described by placing the same curve directly in front of the x -axis — joined by straight lines. The other sides of the solid are bounded planes slicing perpendicularly in a variety of directions.

Figure 10-4:
A solid based on two exponential curves in space.



Notice that when you slice this solid perpendicular with the x -axis, its cross section is always an isosceles right triangle. This is an easy shape to measure, so the slicing method works nicely to measure the volume of this solid. Here are the steps:

1. Find an expression that represents the area of a random cross section of the solid.

The triangle on the y -axis has a height and base of 1 — that is, e^0 . And the triangle on the line $x = 1$ has a height and base of e^1 , which is e . In general, the height and base of any cross section triangle is e^x .

So, here's how to use the formula for the area of a triangle to find the area of a cross section in terms of x :

$$A = \frac{1}{2}b \cdot h = \frac{1}{2}e^x \cdot e^x = \frac{1}{2}e^{2x}$$

2. Use this expression to build a definite integral that represents the volume of the solid.

Now that you know how to measure the area of a cross section, integrate to add all the cross sections from $x = 0$ to $x = 1$:

$$V = \int_0^1 \frac{1}{2} e^{2x} dx$$

3. Evaluate this integral to find the volume.

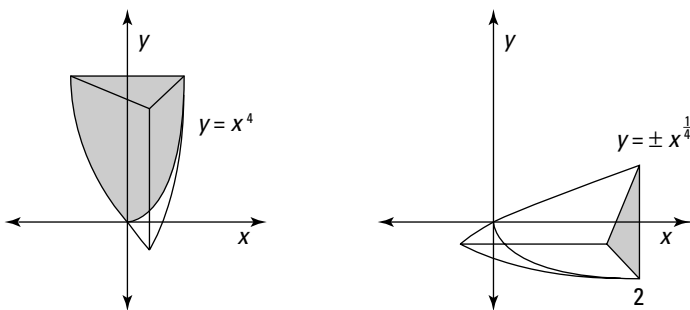
$$\begin{aligned} &= \frac{1}{2} \int_0^1 e^{2x} dx \\ &= \frac{1}{4} e^{2x} \Big|_{x=0}^{x=1} \\ &= \frac{1}{4} e^2 - \frac{1}{4} e^0 \\ &\approx 1.597 \end{aligned}$$

Turning a Problem on Its Side

When using a real meat slicer, you need to find a way to turn whatever you're slicing on its side so that it fits. The same is true for calculus problems.

For example, suppose that you want to measure the volume of the solid shown in Figure 10-5.

Figure 10-5:
Using inverses to get a problem ready for the meat-slicer method.



The good news is that this solid has cross sections that are all similar triangles, so the meat-slicer method will work. Unfortunately, as the problem currently stands, you'd have to make your slices perpendicular to the y -axis. But to use the meat-slicer method, you must make your slices perpendicular to the x -axis.

To solve the problem, you first need to flip the solid over to the x -axis, as shown on the right side of Figure 10-5. The easiest way to do this is to use the inverse of the function $y = x^4$. To find the inverse, switch x and y in the equation and solve for y :

$$x = y^4$$

$$\pm(x^{\frac{1}{4}}) = y$$



Note that the resulting equation $\pm(x^{\frac{1}{4}}) = y$ in this case isn't a function of x because a single x -value can produce more than one y -value. However, you can use this equation in conjunction with the meat-slicer method to find the volume that you're looking for.

1. Find an expression that represents the area of a random cross section of the solid.

The cross section is an isosceles triangle with a height of 3 and a base of $2x^{\frac{1}{4}}$, so use the formula for the area of a triangle:

$$A = \frac{1}{2}bh = \frac{1}{2}(2x^{\frac{1}{4}})(3) = 3x^{\frac{1}{4}}$$

2. Use this expression to build a definite integral that represents the volume of the solid.

$$V = \int_0^2 3x^{\frac{1}{4}} dx$$

3. Solve the integral.

$$3\left(\frac{4}{5}\right)x^{\frac{5}{4}}\Big|_{x=0}^{x=2}$$

$$\frac{12}{5}x^{\frac{5}{4}}\Big|_{x=0}^{x=2}$$

Now, evaluate this expression:

$$= \frac{12}{5}2^{\frac{5}{4}} - 0$$

$$= \frac{12}{5}32^{\frac{1}{4}}$$

$$\approx 5.7082$$

Two Revolutionary Problems

Calculus professors are always on the lookout for new ways to torture their students. Okay, that's a slight exaggeration. Still, sometimes it's hard to fathom exactly why a problem without much practical use makes the Calculus Hall of Fame.

In this section, I show you how to tackle two problems of dubious practical value (unless you consider the practicality passing Calculus III!). First, I show you how to find the volume of a *solid of revolution*: a solid created by spinning a function around an axis. The meat-slicer method, which I discuss in the previous section, also applies to problems of this kind.

Next, I show you how to find the area of a *surface of revolution*: a surface created by spinning a function around an axis. Fortunately, a formula exists for finding or solving this type of problem.

Solidifying your understanding of solids of revolution

A solid of revolution is created by taking a function, or part of a function, and spinning it around an axis — in most cases, either the x -axis or the y -axis.

For example, the left side of Figure 10-6 shows the function $y = 2 \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$.

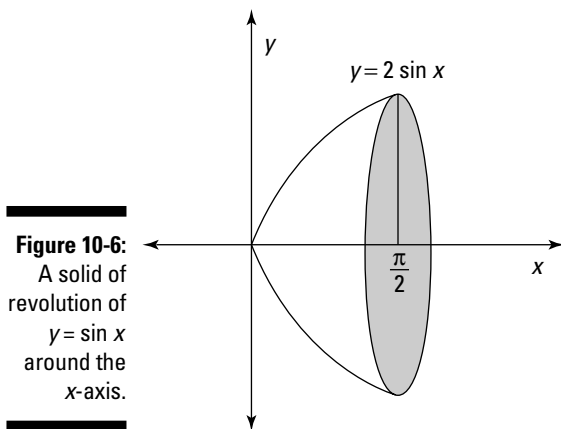


Figure 10-6:
A solid of
revolution of
 $y = \sin x$
around the
 x -axis.



Every solid of revolution has circular cross sections perpendicular to the axis of revolution. When the axis of revolution is the x -axis (or any other line that's parallel with the x -axis), you can use the meat-slicer method directly, as I show you earlier in this chapter.

However, when the axis of revolution is the y -axis (or any other line that's parallel with the y -axis), you need to modify the problem as I show you in the earlier section "Turning a Problem on Its Side."

To find the volume of this solid of revolution, use the meat-slicer method:

- 1. Find an expression that represents the area of a random cross section of the solid (in terms of x).**

This cross section is a circle with a radius of $2 \sin x$:

$$A = \pi r^2 = \pi (2 \sin x)^2 = 4\pi \sin^2 x$$

- 2. Use this expression to build a definite integral (in terms of dx) that represents the volume of the solid.**

This time, the limits of integration are from 0 to $\frac{\pi}{2}$:

$$\begin{aligned} V &= \int_0^{\frac{\pi}{2}} 4\pi \sin^2 x \, dx \\ &= 4\pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \end{aligned}$$

- 3. Evaluate this integral by using the half-angle formula for sines, as I show you in Chapter 7:**

$$\begin{aligned} &= 4\pi \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} \, dx \\ &= 2\pi \left(\int_0^{\frac{\pi}{2}} 1 \, dx - \int_0^{\frac{\pi}{2}} \cos 2x \, dx \right) \\ &= 2\pi \left(x \Big|_{x=0}^{x=\frac{\pi}{2}} - \frac{1}{2} \sin 2x \Big|_{x=0}^{x=\frac{\pi}{2}} \right) \end{aligned}$$

Now, evaluate:

$$\begin{aligned} &= 2\pi \left[\left(\frac{\pi}{2} - 0 \right) - \left(\frac{1}{2} \sin \pi - 0 \right) \right] \\ &= 2\pi \left(\frac{\pi}{2} \right) \\ &= \pi^2 \\ &\approx 9.8696 \end{aligned}$$

So the volume of this solid of revolution is approximately 9.8696 cubic units.

Later in this chapter, I give you more practice measuring the volume of solids of revolution.

Skimming the surface of revolution

The nice thing about finding the area of a surface of revolution is that there's a formula you can use. Memorize it and you're halfway done.

To find the area of a surface of revolution between a and b , use the following formula:

$$A = \int_a^b 2\pi r \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

This formula looks long and complicated, but it makes more sense when you spend a minute thinking about it. The integral is made from two pieces:

- ✓ The arc-length formula, which measures the length along the surface (see Chapter 9)
- ✓ The formula for the circumference of a circle, which measures the length around the surface

So multiplying these two pieces together is similar to multiplying length and width to find the area of a rectangle. In effect, the formula allows you to measure surface area as an infinite number of little rectangles.

When you're measuring the surface of revolution of a function $f(x)$ around the x -axis, substitute $r = f(x)$ into the formula I gave you:

$$A = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

For example, suppose that you want to find the surface of revolution that's shown in Figure 10-7.

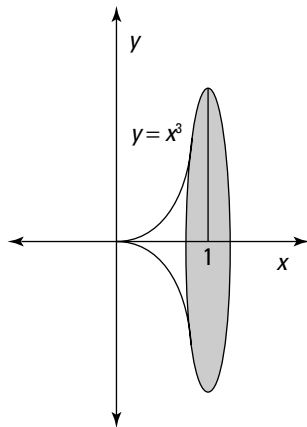


Figure 10-7:
Measuring
the surface
of revolution
of $y = x^3$
between
 $x = 0$ and
 $x = 1$.

To solve this problem, first note that for $f(x) = x^3$, $f'(x) = 3x^2$. So set up the problem as follows:

$$A = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} dx$$

To start off, simplify the problem a bit:

$$= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} dx$$

You can solve this problem by using variable substitution:

$$\begin{aligned} \text{Let } u &= 1 + 9x^4 \\ du &= 36x^3 dx \\ &= \frac{1}{36} \cdot 2\pi \int_1^{10} \sqrt{u} du \end{aligned}$$

Notice that I change the limits of integration: When $x = 0$, $u = 1$. And when $x = 1$, $u = 10$.

$$= \frac{1}{18} \pi \int_1^{10} \sqrt{u} du$$

Now, you can perform the integration:

$$\begin{aligned} &= \frac{1}{18} \pi \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=1}^{u=10} \\ &= \frac{1}{27} \pi u^{\frac{3}{2}} \Big|_{u=1}^{u=10} \end{aligned}$$

Finally, evaluate the definite integral:

$$\begin{aligned} &= \frac{1}{27} \pi 10^{\frac{3}{2}} - \frac{1}{27} \pi 1^{\frac{3}{2}} \\ &= \frac{1}{27} \pi 10 \sqrt{10} - \frac{1}{27} \pi \\ &\approx 3.5631 \end{aligned}$$

Finding the Space Between

In Chapter 9, I show you how to find the area between two curves by subtracting one integral from another. This same principle applies in three dimensions to find the volume of a solid that falls between two different surfaces.

The meat-slicer method, which I describe earlier in this chapter, is useful for many problems of this kind. The trick is to find a way to describe the donut-shaped area of a cross section as the difference between two integrals: one integral that describes the whole shape minus another that describes the hole.

For example, suppose that you want to find the volume of the solid shown in Figure 10-8.

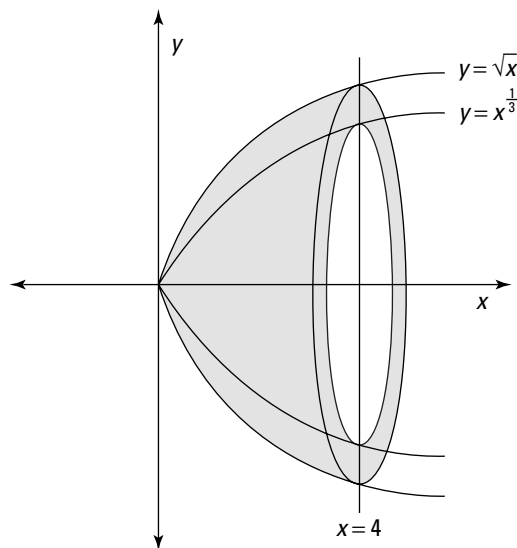


Figure 10-8:
A vase-shaped solid between two surfaces of revolution.

This solid looks something like a bowl turned on its side. The outer edge is the solid of revolution around the x -axis for the function \sqrt{x} . The inner edge is the solid of revolution around the x -axis for the function $x^{\frac{1}{3}}$.

1. Find an expression that represents the area of a random cross section of the solid.

That is, find the area of a circle with a radius of \sqrt{x} and subtract the area of a circle with a radius of $x^{\frac{1}{3}}$:

$$A = \pi \left(\sqrt{x} \right)^2 - \pi \left(x^{\frac{1}{3}} \right)^2 = \pi \left(x - x^{\frac{2}{3}} \right)$$

2. Use this expression to build a definite integral that represents the volume of the solid.

The limits of integration this time are 0 and 4:

$$V = \int_0^4 \pi \left(x - x^{\frac{2}{3}} \right) dx$$

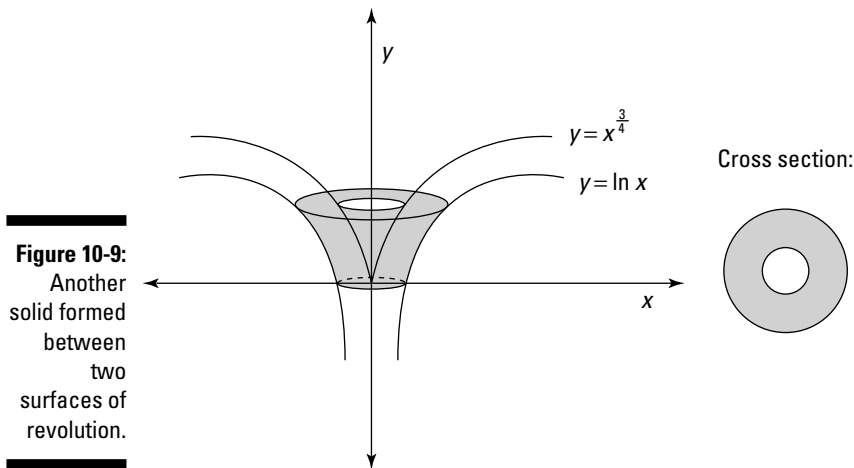
3. Solve the integral:

$$\begin{aligned}
 &= \pi \int_0^4 \left(x - x^{\frac{2}{3}} \right) dx \\
 &= \pi \left(\int_0^4 x \, dx - \int_0^4 x^{\frac{2}{3}} \, dx \right) \\
 &= \pi \left(\frac{1}{2} x^2 \Big|_{x=0}^{x=4} - \frac{3}{5} x^{\frac{5}{3}} \Big|_{x=0}^{x=4} \right)
 \end{aligned}$$

Now, evaluate this expression:

$$\begin{aligned}
 &= \pi \left[\left(\frac{1}{2} 4^2 - 0 \right) - \left(\frac{3}{5} 4^{\frac{5}{3}} - 0 \right) \right] \\
 &= \pi \left(8 - \frac{3}{5} 1,024^{\frac{1}{3}} \right) \\
 &\approx 6.1336
 \end{aligned}$$

Here's a problem that brings together everything you've worked with from the meat-slicer method: Find the volume of the solid shown in Figure 10-9. This solid falls between the surface of revolution $y = \ln x$ and the surface of revolution $y = x^{\frac{3}{4}}$, bounded below by $y = 0$ and above by $y = 1$.

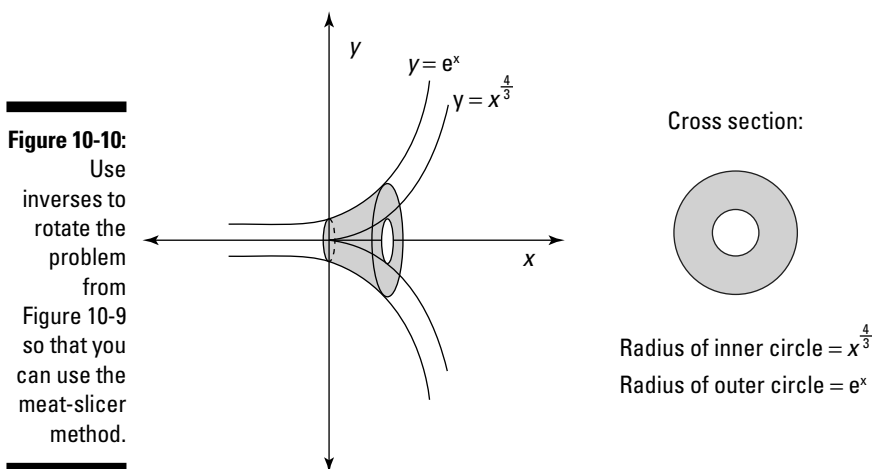


The cross section of this solid is shown in the right side of Figure 10-9: a circle with a hole in the middle.

Notice, however, that this cross section is perpendicular to the y -axis. To use the meat-slicer method, the cross section must be perpendicular to the x -axis. Modify the problem using inverses, as I show you in “Turning a Problem on Its Side”:

$$\begin{aligned} x &= \ln y & x &= y^{\frac{3}{4}} \\ e^x &= y & x^{\frac{4}{3}} &= y \end{aligned}$$

The resulting problem is shown in Figure 10-10.



Now, you can use the meat-slicer method to solve the problem:

- 1. Find an expression that represents the area of a random cross section of the solid.**

That is, find the area of a circle with a radius of e^x and subtract the area of a circle with a radius of $x^{\frac{4}{3}}$. This is just geometry, but I take it slowly so that you can see all the steps. Remember that the area of a circle is πr^2 :

$$\begin{aligned} A &= \text{Area of outer circle} - \text{Area of inner circle} \\ &= \pi (e^x)^2 - \pi \left(x^{\frac{4}{3}}\right)^2 \\ &= \pi e^{2x} - \pi x^{\frac{8}{3}} \end{aligned}$$

- 2. Use this expression to build a definite integral that represents the volume of the solid.**

The limits of integration are 0 and 1:

$$V = \int_0^1 \left(\pi e^{2x} - \pi x^{\frac{8}{3}} \right) dx$$

- 3. Evaluate the integral:**

$$\begin{aligned} &= \int_0^1 \pi e^{2x} dx - \int_0^1 \pi x^{\frac{8}{3}} dx \\ &= \frac{\pi}{2} e^{2x} \Big|_{x=0}^{x=1} - \frac{3\pi}{11} x^{\frac{11}{3}} \Big|_{x=0}^{x=1} \\ &= \left(\frac{\pi}{2} e^2 - \frac{\pi}{2} e^0 \right) - \left(\frac{3\pi}{11} (1)^{\frac{11}{3}} - \frac{3\pi}{11} (0)^{\frac{11}{3}} \right) \\ &= \frac{\pi}{2} e^2 - \frac{\pi}{2} - \frac{3\pi}{11} \\ &\approx 2.9218 \end{aligned}$$

So the volume of this solid is approximately 2.9218 cubic units.

Playing the Shell Game

The *shell method* is an alternative to the meat-slicer method, which I discuss earlier in this chapter. It allows you to measure the volume of a solid by measuring the volume of many concentric surfaces of the volume, called “shells.”

Although the shell method works only for solids with circular cross sections, it’s ideal for solids of revolution around the y -axis, because you don’t have to use inverses of functions, as I show you in “Turning a Problem on Its Side.” Here’s how it works:

- 1. Find an expression that represents the area of a random shell of the solid in terms of x .**
- 2. Use this expression to build a definite integral (in terms of dx) that represents the volume of the solid.**
- 3. Evaluate this integral.**

As you can see, this method resembles the meat-slicer method. The main difference is that you’re measuring the area of shells instead of cross sections.

Peeling and measuring a can of soup

You can use a can of soup — or any other can that has a paper label on it — as a handy visual aid to give you insight into how the shell method works. To start out, go to the pantry and get a can of soup.

Suppose that your can of soup is industrial size, with a radius of 3 inches and a height of 8 inches. You can use the formula for a cylinder to figure out its volume as follows:

$$V = A_b \cdot h = 3^2\pi \cdot 8 = 72\pi$$

Another option is the meat-slicer method, as I show you earlier in this chapter. A third option, which I focus on here, is the shell method.

To understand the shell method, slice the can's paper label vertically, and carefully remove it from the can, as shown in Figure 10-11. (While you're at it, take a moment to read the label so that you're not left with "mystery soup.")

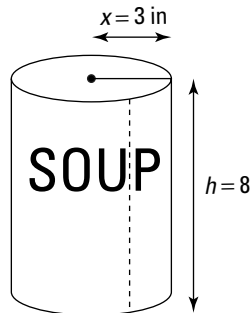
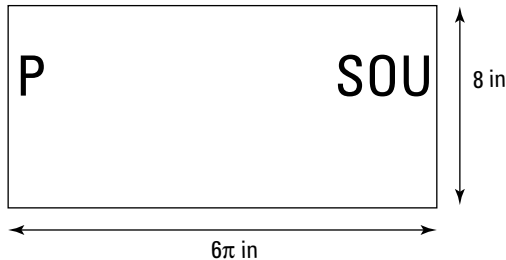


Figure 10-11:

Removing the label from a can of soup can help you understand the shell method.



Notice that the label is simply a rectangle. Its shorter side is equal in length to the height of the can (8 inches) and its longer side is equal to the circumference ($2\pi \cdot 3$ inches = 6π inches). So the area of this rectangle is 48 square inches.

Now here's the crucial step: Imagine that the entire can is made up of infinitely many labels wrapped concentrically around each other, all the way to its core. The area of each of these rectangles is:

$$A = 2\pi x \cdot 8 = 16\pi x$$

The variable x in this case is any possible radius, from 0 (the radius of the circle at the very center of the can) to 3 (the radius of the circle at the outer edge). Here's how you use the shell method, step by step, to find the volume of the can:

- 1. Find an expression that represents the area of a random shell of the can (in terms of x):**

$$A = 2\pi x \cdot 8 = 16\pi x$$

- 2. Use this expression to build a definite integral (in terms of dx) that represents the volume of the can.**

Remember that with the shell method, you're adding up all the shells from the center (where the radius is 0) to the outer edge (where the radius is 3). So use these numbers as the limits of integration:

$$V = \int_0^3 16\pi x \, dx$$

- 3. Evaluate this integral:**

$$\begin{aligned} &= 16\pi \cdot \frac{1}{2} x^2 \Big|_{x=0}^{x=3} \\ &= 8\pi x^2 \Big|_{x=0}^{x=3} \end{aligned}$$

Now, evaluate this expression:

$$= 8\pi (3)^2 - 0 = 72\pi$$

The shell method verifies that the volume of the can is 72π cubic inches.

Using the shell method

One advantage of the shell method over the meat-slicer method comes into play when you're measuring a volume of revolution around the y -axis.

Earlier in this chapter I tell you that the meat-slicer method works best when a solid is on its side — that is, when you can slice it perpendicular to the x -axis. But when the similar cross sections of a solid are perpendicular to the y -axis, you need to use inverses to realign the problem before you can start slicing. (See the earlier section “Turning a Problem on Its Side” for more details.)

This realignment step isn’t necessary for the shell method. This makes the shell method ideal for measuring solids of revolution around the y -axis. For example, suppose that you want to measure the volume of the solid shown in Figure 10-12.

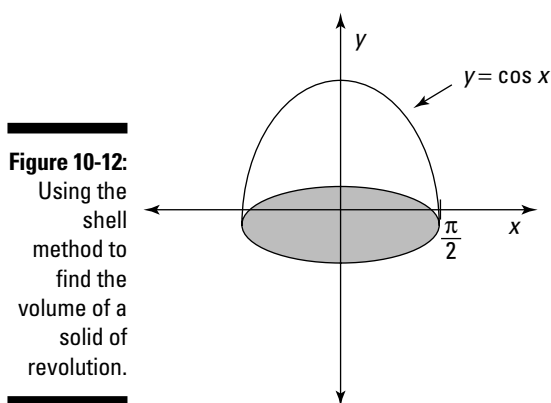


Figure 10-12:
Using the shell method to find the volume of a solid of revolution.

Here’s how the shell method can give you a solution without using inverses:

- 1. Find an expression that represents the area of a random shell of the solid (in terms of x).**

Remember that each shell is a rectangle with two different sides: One side is the height of the function at x — that is, $\cos x$. The other is the circumference of the solid at x — that is, $2\pi x$. So, to find the area of a shell, multiply these two numbers together:

$$A = 2\pi x \cos x$$

- 2. Use this expression to build a definite integral (in terms of dx) that represents the volume of the solid.**

In this case, remember that you’re adding up all the shells from the center (at $x = 0$) to the outer edge (at $x = \frac{\pi}{2}$).

$$\begin{aligned} V &= \int_0^{\frac{\pi}{2}} 2\pi x \cos x \, dx \\ &= 2\pi \int_0^{\frac{\pi}{2}} x \cos x \, dx \end{aligned}$$

3. Evaluate the integral.

This integral is pretty easy to solve using integration by parts:

$$x \sin x + \cos x \Big|_{x=0}^{x=\frac{\pi}{2}}$$

Now, evaluate this expression:

$$\begin{aligned} &= \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \right) - (0 \sin 0 + \cos 0) \\ &= \left(\frac{\pi}{2} + 0 \right) - (0 + 1) \\ &= \frac{\pi}{2} - 1 \\ &\approx 0.5708 \end{aligned}$$

So the volume of the solid is approximately 0.5708 cubic units.

Knowing When and How to Solve 3-D Problems

Because students are so often confused when it comes to solving 3-D calculus problems, here's a final perspective on all the methods in this chapter, and how to choose among them.

First, remember that every problem in this chapter falls into one of these two categories:

- ✓ Finding the area of a surface of revolution
- ✓ Finding a volume of a solid

In the first case, use the formula I provide earlier in this chapter, in “Skimming the surface of revolution.”

In the second case, remember that the key to measuring the volume of any solid is to slice it up in the direction where it has *similar cross sections* whose area can be measured easily — for example, a circle, a square, or a triangle. So, your first question is whether these similar cross sections are arranged horizontally or vertically.

- ✓ **Horizontally** means that the solid is already in position for the meat-slicer method. (If it's helpful, imagine slicing a salami in a meat-slicer. The salami must be aligned lying on its side — that is, horizontally — before you can begin slicing.)
- ✓ **Vertically** means that the solid is standing upright so that the slices are stacked on top of each other.

When the cross sections are arranged horizontally, the meat-slicer method is the easiest way to handle the problem (see “Slicing Your Way to Success” earlier in this chapter).

When the cross sections are arranged vertically, however, your next question is whether these cross sections are circles:

- ✔ If the cross sections are *not* circles, you must use inverses to flip the solid in the horizontal direction (as I discuss in “Turning a Problem on Its Side”).
- ✔ If they *are* circles, you can either use inverses to flip the solid in the horizontal direction (as I discuss in “Turning a Problem on Its Side”) or use the shell method (as I discuss in “Playing the Shell Game”).

